Semiring Rank versus Column Rank

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ABSTRACT

This paper concerns two notions of rank of matrices over semirings: semiring rank and column rank. These two rank functions are the same over fields and Euclidean rings, but differ for matrices over many combinatorially interesting semirings including the nonnegative integer matrices, the fuzzy matrices, and the binary Boolean matrices. We investigate the largest value of r for which the column rank and semiring rank of all $m \times n$ matrices over a given semiring are both r. This value is determined for the semirings mentioned above as well as many others.

1. INTRODUCTION

A semiring is essentially a ring in which only the zero is required to have an additive inverse (a formal definition is given in Section 2). Thus all rings are semirings. So are such combinatorially interesting systems as the Boolean algebra of subsets of a finite set (with addition being union and multiplication being intersection) and the nonnegative integers (with the usual arithmetic). The concepts of matrix theory are defined over a semiring as over a field. Recently a number of authors have studied various problems of semiring matrix theory. In particular, Kim [5] has written an encyclopedic work on

52 Vanderbilt Ave., New York, NY 10017 0024-3795/88/$3.50

matrices over the 2-element Boolean algebra $\mathbb{B}$ of subsets of a singleton. There $\mathbb{B} = \{0,1\}$, where $0 = 0+0 = 1\cdot 0 = 0\cdot 1$ and $1 = 1\cdot 1 = 1+1 = 1+0 = 0+1$. Fuzzy matrices provide another popular example of matrices over a semiring. In that case, the semiring $\mathbb{K}$ of scalars consists of the real numbers $0 \leqslant x \leqslant 1$ with $x + y = \max(x, y)$ and $xy = \min(x, y)$.

This paper is concerned with two notions of rank that arise naturally in matrix theory over semirings. The two rank functions are equal when $\mathbb{S}$ is a field. But they may differ over other semirings.

Let A be an $m \times n$ matrix over $\mathbb{S}$. If $A \neq 0$, then the *rank* of A, $r_{\mathbb{S}}(A)$, is the least k for which there exist $m \times k$ and $k \times n$ matrices F and G over $\mathbb{S}$ such that $A = FG$; $r_{\mathbb{S}}(0) = 0$. When $\mathbb{S}$ is a field, then $r_{\mathbb{S}}$ is the usual rank function. Kim [5] calls $r_{\mathbb{B}}$ the *Schein rank*.

The concepts of "dimension" and "column space" are defined (see Section 2) so as to coincide with the familiar definitions when $\mathbb{S}$ is a field. Then we can define the column rank of A, $c_{\mathbb{S}}(A)$, as the dimension of the column space of A. It follows that $0 \leqslant r_{\mathbb{S}}(A) \leqslant c_{\mathbb{S}}(A) \leqslant n$ for all $m \times n$ matrices A over $\mathbb{S}$.

The column rank of a matrix may actually exceed its rank over some semirings. For example, the column rank of

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

is 4 over $\mathbb{B}$ (see Corollary 2.5.2 below), despite the fact that its rank over $\mathbb{B}$ cannot exceed 3, its number of rows, by the definition of rank.

Let $\mu(\mathbb{S}, m, n)$ be the largest integer k such that for all $m \times n$ matrices A over $\mathbb{S}$, $r_{\mathbb{S}}(A) = c_{\mathbb{S}}(A)$ if $r_{\mathbb{S}}(A) \leqslant k$. The previous example shows that $\mu(\mathbb{B}, 3, 4) \leqslant 2$. In general $0 \leqslant \mu(\mathbb{S}, m, n) \leqslant \min(m, n)$.

In Theorems 1, 2, 3, and 4 we determine μ for a large variety of semirings. These results enable us to compute μ for such semirings as the nonnegative integers $\mathbb{Z}^+$, the nonnegative rationals $\mathbb{Q}^+$, the 2-element Boolean algebra $\mathbb{B}$, the fuzzy scalars $\mathbb{K}$, and many others. Sample results:

$$\mu(\mathbb{Z}^+, m, n) = \begin{cases} 1 & \text{if } \; m \geqslant 1 \text{ and } n = 1, \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$

$$\mu(\mathbb{Q}^+, m, n) = \begin{cases} 1 & \text{if } \; \min(m, n) = 1, \\ 3 & \text{if } \; m \geqslant 3 \text{ and } n = 3, \\ 2 & \text{otherwise}, \end{cases} \tag{2}$$

$$\mu(\mathbb{B}, m, n) = \begin{cases} 1 & \text{if} \quad \min(m, n) = 1, \\ 3 & \text{if} \quad m \geqslant 3 \text{ and } n = 3, \\ 2 & \text{otherwise,} \end{cases} \tag{3}$$

$$\mu(\mathbb{K}, m, n) = \begin{cases} 2 & \text{if} \quad m \geqslant 2 \text{ and } n = 2, \\ 1 & \text{otherwise.} \end{cases} \tag{4}$$

2. PRELIMINARIES

A *semiring* (see e.g. Gregory and Pullman [4] or Kim [5]) consists of a set $\mathbb{S}$ and two binary operations on $\mathbb{S}$, addition and multiplication, such that:

(1) $\mathbb{S}$ is an Abelian monoid under addition (identity denoted by 0);
(2) $\mathbb{S}$ is a monoid under multiplication (identity denoted by 1);
(3) multiplication distributes over addition; and
(4) $s0 = 0s = 0$ for all $s \in \mathbb{S}$.

Usually $\mathbb{S}$ denotes both the semiring and the set. When some confusion could arise, we denote the semiring by e.g. $(\mathbb{S}, +, \times)$, if addition is denoted $+$ and multiplication $\times$. If $(\mathbb{S}, \times)$ is Abelian, we say $\mathbb{S}$ is *commutative*.

Let $\mathbb{S}$ be any set of two or more elements. If $\mathbb{S}$ is totally ordered by $<$, that is, $\mathbb{S}$ is a chain under $<$ (i.e., $x < y$ or $y < x$ for all distinct x, y in $\mathbb{S}$), then define $x + y$ as $\max(x, y)$ and xy as $\min(x, y)$ for all x, y in $\mathbb{S}$. If $\mathbb{S}$ has a universal lower bound and a universal upper bound, then $\mathbb{S}$ becomes a semiring: a *chain* semiring.

Let $\mathbb{H}$ be any nonempty family of sets nested by inclusion, $0 = \bigcap_{x \in \mathbb{H}} x$, and $1 = \bigcup_{x \in \mathbb{H}} x$. Then $\mathbb{S} = \mathbb{H} \cup \{0, 1\}$ is a chain semiring.

Let α, ω be real numbers with $\alpha < \omega$. Define $\mathbb{S} = \{\beta \in \mathbb{R} : \alpha \leqslant \beta \leqslant \omega\}$. Then $\mathbb{S}$ is a chain semiring with $\alpha =$ "0" and $\omega =$ "1." It is isomorphic to the chain semiring in the previous example with $\mathbb{H} = \{[\alpha, \beta] : \alpha \leqslant \beta \leqslant \omega\}$.

If in particular we choose the real numbers 0 and 1 as α and ω in the previous example, then the system of $m \times n$ matrices over $\mathbb{K} \equiv \{\beta : 0 \leqslant \beta \leqslant 1\}$ is the fuzzy matrices.

If we take $\mathbb{H}$ to be a singleton, say $\{a\}$, and denote $\varnothing$ by 0 and $\{a\}$ by 1, the resulting chain semiring (called $\mathbb{B}$) is a subsemiring of every chain semiring.

The set of $m \times n$ matrices with entries in a semiring $\mathbb{S}$ is denoted by $\mathcal{M}_{m,n}(\mathbb{S})$. The $m \times n$ zero matrix $O_{m,n}$ and the $n \times n$ identity matrix I_n are defined as if $\mathbb{S}$ were a field. Addition, multiplication by scalars, and the product of matrices are also defined as if $\mathbb{S}$ were a field. Thus $\mathcal{M}_{n,n}(\mathbb{S})$ is a semiring under matrix addition and multiplication. If $\mathbb{S}$ is not commutative, unless otherwise indicated we'll take the operation of multiplication by scalars to be left multiplication: $(s, \mathbf{a}) \to s\mathbf{a}$.

If $\mathscr{V}$ is a nonempty subset of $\mathbb{S}^k \equiv \mathscr{M}_{k,1}(\mathbb{S})$ that is closed under addition and multiplication by scalars, then $\mathscr{V}$ is called a *vector space* over $\mathbb{S}$. The notions of subspace and of spanning or generating sets are the same as if $\mathbb{S}$ were a field.

We'll use the notation $\langle \mathscr{F} \rangle$ to denote the subspace spanned by the subset $\mathscr{F}$ of $\mathscr{V}$. As with fields, a basis for a vector space $\mathscr{V}$ is a generating subset of least cardinality. That cardinality is the *dimension*, $\dim(\mathscr{V})$, of $\mathscr{V}$.

The *column rank* $c(A) = c_{\mathbb{S}}(A)$ of an $m \times n$ matrix A over $\mathbb{S}$ is the dimension of the space $\langle A \rangle$ spanned by its columns. It follows directly from the definitions that for all $m \times n$ matrices A over $\mathbb{S}$:

(2.1) $0 \leqslant c(A) \leqslant n$;
(2.2) $c(B) \leqslant c(A)$ if B is obtained by deleting some rows of A.

We shall see later on that when $\mathbb{S}$ is not a field, we can have $c(A) > r(A)$, $c(A) > m$, $c(A) \neq c(A^{\mathsf{T}})$, and $c(B) > c(A)$ for some submatrix B of A.

Here is a somewhat better-behaved notion of rank for semirings. Define the *rank* of a nonzero $m \times n$ matrix A over $\mathbb{S}$ as the least integer k such that $A = BC$ for some $m \times k$ and $k \times n$ matrices B and C over $\mathbb{S}$. The rank of $O_{m,n}$ is 0.

We denote the rank of A by $r(A)$ or by $r_{\mathbb{S}}(A)$. In [5] Kim calls $r_{\mathbb{B}}(A)$ the *Schein rank* of A. In [4] $r_{\mathbb{S}}(A)$ is called the *semiring rank* of A.

Here are some properties of rank that stem directly from the definitions. For all $m \times n$ matrices A over $\mathbb{S}$:

(i) $0 \leqslant r(A) \leqslant \min(m, n)$,
(ii) $r(B) \leqslant r(A)$ for all submatrices B of A, and
(iii) $r(A) = r(A^{\mathsf{T}})$.

Suppose A, B, and C are $m \times n$, $m \times k$ matrices over $\mathbb{S}$. If $A = BC$, we say that B is a *left divisor* and C is a *right divisor* of A. If A is a (left) divisor of B and B is a (left) divisor of A, we say that A is a (*left*) *associate* of B. *Right associate* is defined symmetrically.

LEMMA 2.1. *The matrices A and B have the same column space if and only if they are left associates.*

LEMMA 2.2.

(a) *The rank of any nonzero matrix is the minimum number of columns in its left divisors.*

(b) *The column rank of any nonzero matrix is the minimum number of columns in its left associates.*

LEMMA 2.3. $c(A) \geqslant r(A)$ *for all* $A \in \mathscr{M}_{m,n}(\mathbb{S})$.

LEMMA 2.4. $c(A) = \min\{r(X) : AX = A\}$ *for all* $A \in \mathscr{M}_{m,n}(\mathbb{S})$.

Proof. Let $l = \min\{r(X) : AX = A\}$ and $k = c(A)$. Then A has a left associate B, which is $m \times k$ by Lemma 2.2(b). Then $A = BC$ and $B = AD$ for some C and D, so $A = ADC$. Hence $r(DC) \geqslant l$. But $r(DC) \leqslant k$ because D has k columns. Thus $k \geqslant l$. Choose X so that $l = r(X)$ and $AX = A$. By the definition of r, $X = FG$ where F has l columns. Then AF has l columns and is a left associate of A, so $k \leqslant l$ by Lemma 2.2. ■

A set $\mathscr{A}$ of vectors over $\mathbb{S}$ is *linearly dependent* if for some $\mathbf{a} \in \mathscr{A}$, $\mathbf{a} \in \langle A \setminus \{a\} \rangle$. Otherwise $\mathscr{A}$ is *linearly independent*. If $a = b + x$ for some $x \in \mathbb{S}$, we write $a \geqslant b$. The relation $\geqslant$ is extendable entrywise to vectors and matrices.

LEMMA 2.5. *Suppose* S *is* antinegative (*that is, only* 0 *has an additive inverse*), $\mathscr{A}$ *and* $\mathscr{B}$ *are sets of vectors in* $\mathbb{S}^k$ $(= \mathscr{M}_{k,1}(\mathbb{S}))$, *and* $\mathscr{A}$ *is independent. Then* $\langle \mathscr{A} \rangle = \langle \mathscr{B} \rangle$ *implies that for all* $\mathbf{a} \in \mathscr{A}$, *there exists* $\mathbf{b} \in \mathscr{B}$ *and there exist nonzero scalars* σ *and* τ *such that* $\sigma\tau \neq 0$, $\mathbf{a} \geqslant \sigma\mathbf{b}$, *and* $\mathbf{b} \geqslant \tau\mathbf{a}$.

Proof. Since $\mathbb{B}^k$ is finite, we have $\mathscr{A} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p\}$ and $\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_q\}$. We may assume $\mathscr{A} \neq \varnothing$ and no $\mathbf{b}_i = \mathbf{0}$. Let $t \leqslant p$. Then for some scalars β_i and α_{ij}

$$\mathbf{a}_t = \sum_{i=1}^{q} \beta_i \mathbf{b}_i \quad \text{and} \quad \mathbf{b}_i = \sum_{j=1}^{p} \alpha_{ij} \mathbf{a}_j.$$

Now $\sum_{i=1}^{q} \beta_i \alpha_{it} \neq 0$ because $\mathscr{A}$ is independent. So for some l, we have $\beta_l \alpha_{lt} \neq 0$, $\mathbf{a}_t \geqslant \beta_l \mathbf{b}_l$, and $\mathbf{b}_l \geqslant \alpha_{lt} \mathbf{a}_t$. ■

COROLLARY 2.5.1. *Every subspace* $\mathscr{V}$ *of* $\mathbb{B}^k$ *has a unique basis: a maximum independent subset of* $\mathscr{V}$.

COROLLARY 2.5.2. *If the columns of $A \in \mathscr{M}_{m,n}(\mathbb{B})$ are linearly independent, then $c(A) = n$.*

EXAMPLE 2.1. Let $\mathbb{S}$ be the Boolean algebra of subsets of a two element set with singletons p and q. Then $\mathbb{S} = \{0, p, q, 1\}$, $p + q = 1$, and $pq = 0$. Let $\mathscr{V}$ be the subspace of $\mathbb{S}^2$ spanned by

$$\mathscr{A} = \left\{ \begin{bmatrix} p \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ q \end{bmatrix} \right\}.$$

Then

$$\mathscr{V} = \left\langle \begin{bmatrix} p \\ q \end{bmatrix} \right\rangle$$

because

$$\begin{bmatrix} p \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ q \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}, \quad p\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} p \\ 0 \end{bmatrix}, \quad \text{and} \quad q\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ q \end{bmatrix}.$$

Hence

$$c\left(\begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} \right) = 1$$

even though $\mathscr{A}$ is independent and $\mathbb{S}$ is antinegative. Thus

$$A = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$$

is an example of a matrix with less than full column rank whose columns are linearly independent.

LEMMA 2.6. *If B is obtained by deleting some rows of A, then $c(B) \leqslant c(A)$.*

Proof. For some $n \times n$ matrix X, we have $AX = A$ and $r(X) = c(A)$, by Lemma 2.4. Let U be the $(m-1) \times m$ matrix obtained by deleting row i from I_m. Then $(UA)X = UA$, the matrix obtained by deleting row i from A.

It follows from Lemma 2.4 that $r(X) \geqslant c(UA)$. Hence $c(A) \geqslant c(UA)$. The rest follows by induction on the number of rows deleted. ∎

EXAMPLE. 2.2. Let

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

and $\mathbb{S} = \mathbb{B}$, the 2-element Boolean algebra. Then $c(A) = 3$ because the last three columns are independent and span the column space of A. If we delete column 5 from A to obtain A', then $c(A') = 4$ by Corollary 2.5.1.

Thus suppressing a column may increase the column rank, even though suppressing a row can at worst reduce it.

Example 2.1 shows that $r\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is not necessarily $r(A) + r(B)$. We do have the following corollary to Lemma 2.6 and property (ii) of rank.

COROLLARY 2.6.1. *For any $p \times q$ matrix A over $\mathbb{S}$, the rank of $\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$ is $r(A)$ and its column rank is $c(A)$.*

LEMMA 2.7. *Over any semiring $\mathbb{S}$, if $c(A) > r(A)$ for some $p \times q$ matrix A, then for all $m \geqslant p$ and $n \geqslant q$,*

$$\mu(\mathbb{S}, m, n) < r(A).$$

Proof. Follows directly from the definition of μ and Corollary 2.6.1. ∎

3. THE VALUES OF μ

3.1. Principal Semirings

A semiring $\mathbb{S}$ is *principal* if every nontrivial subspace of $\mathbb{S}$ has dimension 1. Here $\mathbb{S}$ is thought of as a vector space of 1-tuples. As we shall see soon, chain semirings and Boolean algebras are principal semirings. So are principal ideal domains. On the other hand $\mathbb{Z}^+$, the semiring of nonnegative integers, is not a principal semiring, because, for example, the subspace $\mathscr{V}$ of $\mathbb{Z}^+$

generated by $\{2,3\}$ is given by $\mathscr{V} = \{0,2,3,4,5,\dots\}$, which is evidently not the set of multiples of any of its elements.

LEMMA 3.1. *$\mathbb{S}$ is a principal semiring if and only if every 1×2 nonzero matrix over $\mathbb{S}$ has column rank* 1.

Proof. The condition is obviously necessary. Sufficiency is proved by induction. ∎

EXAMPLE 3.1. If $\mathbb{S}$ is a chain semiring or a Boolean algebra, then $\mathbb{S}$ is principal, because if we let $A=[a,b]$, then $\langle A\rangle = \langle a+b\rangle$, and hence $c(A)=1$ unless both a and b are 0.

THEOREM 1. *Suppose* $\min(m,n)>1$. *Then*

$$\mu(\mathbb{S},m,n)=0$$

if and only if

$$\mathbb{S} \textit{ is not a principal semiring}$$

if and only if

$$c[a,b]=2 \qquad \textit{for some} \quad a,b\in\mathbb{S}.$$

Proof. In view of Lemma 3.1 it is enough to show that $\mu>0$ when $\mathbb{S}$ is a principal semiring. Let $\mathbb{S}$ be such a semiring and A be an arbitrary matrix of rank 1 over $\mathbb{S}$. Then $A=\mathbf{a}\mathbf{b}^{\mathsf{T}}$ for some column vectors $\mathbf{a}$, $\mathbf{b}$ and $a_ib_j\neq 0$ for some i and j. Let $\mathscr{V}$ be the row space of A^{T}, that is, $\mathscr{V} = \langle b_1\mathbf{a}^{\mathsf{T}}, b_2\mathbf{a}^{\mathsf{T}},\dots,b_n\mathbf{a}^{\mathsf{T}}\rangle$. But

$$\langle b_1,b_2,\dots,b_n\rangle=\langle\gamma\rangle \qquad \text{for some} \quad \gamma\in\mathbb{S}, \tag{3.1}$$

because $\mathbb{S}$ is a principal semiring. Let $s_1,s_2,\dots,s_n$ be arbitrary scalars. Then there exist scalars x_i such that $\Sigma s_ib_i\mathbf{a}^{\mathsf{T}}=\Sigma s_ix_i\gamma\mathbf{a}^{\mathsf{T}}$ and hence $\mathscr{V}\subseteq\langle\gamma\mathbf{a}^{\mathsf{T}}\rangle$. Again by (3.1), there exist scalars y_i such that $\gamma=\Sigma y_ib_i$ and hence $\gamma\mathbf{a}^{\mathsf{T}}\in\mathscr{V}$. Thus $\mathscr{V}=\langle\gamma\mathbf{a}^{\mathsf{T}}\rangle$ and hence $\dim(\mathscr{V})=1$. It follows that $c(A)=1$. ∎

COROLLARY 3.1. *Unless $n=1$, $\mu(\mathbb{Z}^+,m,n)=0$.*

COROLLARY 3.2. *For all m,n, $\mu(\mathbb{Z},m,n)\geqslant 1$.*

COROLLARY 3.3. *If $\mathbb{S}$ is a chain semiring or a subsemiring of a Boolean algebra, then $\mu(\mathbb{S}, m, n) \geqslant 1$ for all m, n.*

Proof. Apply Example 3.1 and Theorem 1. ■

3.2. *Chain Semirings*

Let $\mathbb{S}$ be any semiring and $\mathbb{B}$ be the Boolean algebra of two elements. For each $x \in \mathbb{S}$ let $\bar{x}$, its *pattern*, be 1 if $x \neq 0$ and 0 otherwise. Then $x \to \bar{x}$, the *pattern mapping*, maps $\mathbb{S}$ into $\mathbb{B}$. If $\mathbb{S}$ is antinegative, then the pattern mapping induces a homomorphism of $(\mathbb{S}, +)$ to $(\mathbb{B}, +)$. If $\mathbb{S}$ has no zero divisors, then it induces a homomorphism of $(\mathbb{S}, \times)$ to $(\mathbb{B}, \times)$. If $A \in \mathcal{M}_{m,n}(\mathbb{S})$, define $\bar{A}$, the *pattern* of A, to be $[\bar{a}_{ij}]$, the $m \times n$ matrix of patterns of the entries of A. Then the mapping $\mathcal{M}_{m,n}(\mathbb{S}) \to \mathcal{M}_{m,n}(\mathbb{B})$ is a semiring homomorphism (the *pattern homomorphism*) if $\mathbb{S}$ is antinegative and zero-divisor-free.

LEMMA 3.2.1. *If $\mathbb{S}$ is a zero-divisor-free, antinegative semiring, then $c(A) \geqslant c(\bar{A})$ for all $A \in \mathcal{M}_{m,n}(\mathbb{S})$.*

Proof. Suppose $c(A) = k$, and B is the $m \times k$ associate of A ensured by Lemma 2.2. Then $\bar{B}$ is an associate of $\bar{A}$, since the pattern mapping is a homomorphism under our assumptions about $\mathbb{S}$. Then by Lemma 2.2 applied to $\mathbb{B}$, $c(\bar{A}) \leqslant k$. ■

COROLLARY 3.2.1. *If $\mathbb{S}$ is a zero-divisor-free, antinegative semiring, and $A \in \mathcal{M}_{m,n}(\mathbb{S})$, then $c(A) = n$ if $c(\bar{A}) = n$.*

EXAMPLE 3.2.1. *Let $\mathbb{S} = \mathbb{Z}^+$ and $A = [2,3,5,7,11,\ldots,p_n]$ where p_n is the* n*th prime. Then $c(A) = n$. But $c(\bar{A}) = 1$ by Corollary* 2.5.1. *This shows that strict inequality can hold in Lemma* 3.2.

EXAMPLE 3.2.2. *Let $\mathbb{S} = \{0, p, q, 1\}$ be the Boolean algebra of* 4 *elements and*

$$A = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$$

as in Example 2.1. *Then $c(A) = 1$ because both columns of A are multiples of their sum $\begin{bmatrix} p \\ q \end{bmatrix}$. On the other hand, $c(\bar{A}) = 2$ by Corollary* 2.5.1. *This shows* *the necessity of the no zero divisor condition in the hypothesis of Lemma* 3.2.1.

LEMMA 3.2.2. *If* $\mathbb{C}$ *is any chain semiring and* $m \geqslant 3$ *and* $n > 3$, *then* $\mu(\mathbb{C}, m, n) \leqslant 2$.

Proof. Let

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

Then $c(A) = 4$ by Corollary 3.2.1 and Corollary 2.5.2. Nevertheless, $r(A) \leqslant 3$ by property (i) of rank. The rest follows from Lemma 2.7. ∎

THEOREM 2. *Let* $\mathbb{C}$ *be any chain semiring other than* $\mathbb{B}$. *Then* $\mu(\mathbb{C}, m, n) = 2$ *if* $2 = n \leqslant m$, *and* $\mu(\mathbb{C}, m, n) = 1$ *otherwise.*

Proof. By Example 3.1 and Theorem 1, $\mu(\mathbb{C}, m, n) \geqslant 1$ for all m and n. Choose $p \in \mathbb{C} \setminus \mathbb{B}$, then let

$$A = \begin{bmatrix} p & p & p \\ p & p & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

We have $r(A) \leqslant 2$ because

$$A = \begin{bmatrix} p & 0 \\ 1 & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p & p & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Now the column space of A is

$$\mathscr{V} = \left\{ x \begin{bmatrix} p \\ p \\ 0 \end{bmatrix} + y \begin{bmatrix} p \\ p \\ p \end{bmatrix} + z \begin{bmatrix} p \\ p \\ 1 \end{bmatrix} + w \begin{bmatrix} p \\ 1 \\ 1 \end{bmatrix} \middle| 0 \leqslant x, y \leqslant p, 0 \leqslant z, w \leqslant 1 \right\}.$$

Let $\mathscr{B}$ be any subset of $\mathscr{V}$ generating $\mathscr{V}$. Let

$$\mathbf{a} = \begin{bmatrix} p \\ p \\ 1 \end{bmatrix}.$$

If $\mathbf{a} \notin \mathscr{B}$, then

$$\mathbf{a} = \begin{bmatrix} x+y+wp \\ x+y+w \\ y+w \end{bmatrix} \qquad \text{for some} \quad x, y \leqslant p \text{ and } w.$$

Now $1 = y + w$ and $y \leqslant p$, so that $y + w = w$ and $w = 1$. But then $p = x + y + w = 1$, a contradiction, since $p \in \mathbb{C} \setminus \mathbb{B}$. Hence $\mathbf{a} \in \mathscr{B}$. Similarly,

$$\mathbf{b} = \begin{bmatrix} p \\ 1 \\ 1 \end{bmatrix}$$

is in $\mathscr{B}$. Let

$$\mathbf{c} = \begin{bmatrix} p \\ p \\ 0 \end{bmatrix}.$$

Then $\mathbf{c} \notin \mathscr{B}$ would imply that $c_3 = y + z + w$ for some y, z, and w, one of which is nonzero, which is possible. Hence $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\} \subseteq \mathscr{B}$. Therefore $c(A) = 3$. Consequently $\mu(\mathbb{C}, m, n) \leqslant 1$ when $m \geqslant 3$ and $n \geqslant 3$, by Lemma 2.7. Evidently we may assume that $n > 1$. If $2 = m < n$, let

$$A_0 = \begin{bmatrix} p & p & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Then $r(A_0) = 2$. If $c(A_0) \neq 3$ then $c(A_0) \leqslant 2$. Let B_0 be a left associate of A_0 with $k = c(A_0)$ columns. Then $A = FA_0 = FB_0C$, where

$$F = \begin{bmatrix} p & 0 \\ 1 & p \\ 0 & 1 \end{bmatrix}.$$

So $c(A) \leqslant c(FB_0)$, because $\langle A \rangle \subseteq \langle FB_0 \rangle$. Also $c(FB_0) \leqslant 2$, because B_0 has $c(A_0)$ columns. This contradicts the fact that $c(A) = 3$. Hence $\mu(\mathbb{C}, 2, n) \leqslant 1$. If $1 = m < n$, then the fact that $\langle x_1, x_2, \ldots, x_n \rangle = \langle \Sigma_{i=1}^{n} x_i \rangle$ implies that $\mu(\mathbb{C}, 1, n) = 1$. If $2 = n \leqslant m$, then $c(A) = 2$ whenever $r(A) = 2$ by property (2.1) of column rank and Lemma 2.3. ∎

THEOREM 3.

$$\mu(\mathbb{B}, m, n) = 1 \qquad \textit{whenever } \min(m, n) = 1,$$

$$\mu(\mathbb{B}, m, 3) = 3 \qquad \textit{for all } m \geqslant 3,$$

$$\mu(\mathbb{B}, m, n) = 2 \qquad \textit{otherwise}.$$

Proof. Let

$$A_0 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix};$$

then $r(A_0) \leqslant 3$. We have $c(A_0) = 4$ because the columns of A_0 are independent. By Lemma 2.7, for all $m \geqslant 3$ and $n \geqslant 4$

$$\mu(\mathbb{B}, m, n) \leqslant 2. \tag{3.2}$$

Suppose $m \geqslant 2$ and $n \geqslant 2$, and $A \in \mathscr{M}_{m,n}(\mathbb{B})$. If $r(A) = 2$ then $A = FG$, where F is $m \times 2$ and G is $2 \times n$. For some permutation matrix P,

$$GP = \begin{bmatrix} 1 & 0 & x_3 & \cdots & x_n \\ 0 & 1 & y_3 & \cdots & y_n \end{bmatrix}.$$

Otherwise $r(G) = 1$ and hence $r(A) = 1$, a contradiction. Therefore two columns of F are columns of A that generate the column space of A. It follows that $c(A) = 2$. Therefore

$$\mu(\mathbb{B}, m, n) \geqslant 2 \qquad \text{when} \quad \min(m, n) \geqslant 2, \tag{3.3}$$

$$\mu(\mathbb{B}, 2, n) = 2 \qquad \text{for} \quad n \geqslant 2. \tag{3.4}$$

If $m \geqslant 3$ and $n \geqslant 4$, then $\mu(\mathbb{B}, m, n) = 2$ by the inequalities (3.2) and (3.3).

The inequality (3.3), Lemma 2.3, and property (2.1) of column rank imply that $\mu(\mathbb{B}, m, 3) = 3$ when $m \geqslant 3$ and that $\mu(\mathbb{B}, m, 2) = 2$ when $m \geqslant 2$.

Example 3.1 implies that $\mu(\mathbb{B}, 1, n) = 1$ for all $n \geqslant 1$. Evidently, $\mu(\mathbb{B}, m, 1) = 1$ for all $m \geqslant 1$. Therefore $\mu(\mathbb{B}, m, n) = 1$ whenever $m = 1$ or $n = 1$. ■

THEOREM 4. *If $\mathbb{S}$ is a field or a Euclidean domain, then $\mu(\mathbb{S}, m, n) = \min(m, n)$ for all m and n.*

Proof. Let A be any nonzero matrix over $\mathbb{S}$. Under either hypothesis there exist matrices W and U over $\mathbb{S}$ with inverses over $\mathbb{S}$, and a diagonal $k \times k$ matrix D over $\mathbb{S}$ such that

$$A = W\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}U$$

where no $d_{ii} = 0$.

Let $B = AU^{-1}$. The definition of $r(X)$ implies that it is invariant under pre- or postmultiplication of X by an invertible matrix. Therefore $r(A) = r(B) = r(D)$.

The matrices A and B have the same column space because they are associates. Therefore $c(A) = c(B)$. But $k \geqslant c(B)$ because B has at most k nonzero columns. Hence $k \geqslant c(A)$. But $r(D) = k$ because $\mathbb{S}$ has no zero divisors. Therefore $r(A) = c(A)$. ■

3.3. *The Nonnegative Part of a Real Subfield*

Let $\mathbb{F}$ be a subfield of the reals, and $\mathbb{F}^+$ be the subset of $\mathbb{F}$ consisting of the nonnegative members of $\mathbb{F}$.

LEMMA 3.3. *If A is an $m \times n$ matrix over $\mathbb{F}^+$, whose columns are independent, then $c(A) = n$.*

Proof. We'll use the characterization of $c(A)$ given by Lemma 2.4. Suppose $A = AX$; then the jth column of A is given by

$$\mathbf{a}_j = \sum_{i=1}^{n} x_{ij}\mathbf{a}_i \qquad \text{for all } j. \tag{3.5}$$

Since the columns are independent, no $\mathbf{a}_j = \mathbf{0}$. It follows that all $x_{ij} \neq 0$. Since $\mathbb{F}^+$ is antinegative, it follows that (entrywise) $\mathbf{a}_j \geqslant x_{jj}\mathbf{a}_j$ for all j. But some entry in A is positive, so $a_{tj} \geqslant x_{jj}a_{tj}$ for some $a_{tj} > 0$. Hence

$$1 \geqslant x_{jj} > 0 \qquad \text{for all } j. \tag{3.6}$$

If $x_{jj} < 1$ for some j, then $1 - x_{jj} > 0$. Hence $(1 - x_{jj})^{-1} \in \mathbb{F}^+$. According to Equation (3.5), $\mathbf{a}_j$ would then be a linear combination of $\{\mathbf{a}_i \mid i \neq j\}$.

Hence $x_{jj}=1$. It follows that $\Sigma_{i\neq j}x_{ij}\mathbf{a}_i=\mathbf{0}$. For all t, $\Sigma_{i\neq j}x_{ij}a_{ti}=0$. Summing on t, we get $\Sigma_{i\neq j}x_{ij}\alpha_i=0$, where $\alpha_i=\Sigma_{t=1}^{m}a_{ti}$. But $\alpha_i>0$ because $\mathbf{a}_i\neq\mathbf{0}$. Thus $x_{ij}=0$ for all $i\neq j$. Hence $X=I$. By Lemma 2.4, $c(A)=n$. ■

LEMMA 3.4. *If* $\min(m,n)\geqslant 3$ *and* $n\geqslant 4$, *then* $\mu(\mathbb{F}^+,m,n)\leqslant 2$.

Proof. Let

$$A=\begin{bmatrix}0&1&1&1\\1&0&1&1\\1&1&0&3\end{bmatrix}.$$

Clearly $r(A)\leqslant 3$.

We can show that $c(A)=4$ by applying Lemma 3.3. Therefore $\mu(\mathbb{F}^+,m,n)\leqslant 2$. ■

NOTE. Lemma 3.4 holds for any antinegative semiring which contains a subsemiring isomorphic to $\mathbb{Z}^+$.

THEOREM 5.

$$\mu(\mathbb{F}^+,m,n)=1 \qquad \textit{whenever} \quad \min(m,n)=1,$$

$$\mu(\mathbb{F}^+,m,3)=3 \qquad \textit{for all} \quad m\geqslant 3,$$

$$\mu(\mathbb{F}^+,m,n)=2 \qquad \textit{otherwise}.$$

Proof. If $\min(m,n)=1$, the theorem follows by Theorem 1.

If $n=2$, clearly $c(A)=2$ whenever $r(A)=2$. If $m=2$, $A\in\mathcal{M}_{2,n}(\mathbb{F}^+)$, and $r(A)=2$, then we have two cases.

Case 1: A has a zero entry. Permuting rows and/or columns and multiplying by a diagonal matrix, we can assume that

$$A=\begin{bmatrix}0&0&\cdots&0&1&1&\cdots&1\\1&a_{22}&\cdots&a_{2k}&a_{2,k+1}&a_{2,k+2}&\cdots&a_{2n}\end{bmatrix},$$

where $a_{2n}\leqslant a_{2j}$ for all $j\geqslant k+1$. Now,

$$\begin{bmatrix}0\\a_{2i}\end{bmatrix}=a_{2i}\begin{bmatrix}0\\1\end{bmatrix} \qquad \text{for all} \quad i\leqslant k,$$

and for $j \geqslant k+1$,

$$\begin{bmatrix} 1 \\ a_{2j} \end{bmatrix} = \begin{bmatrix} 1 \\ a_{2n} \end{bmatrix} + (a_{2j} - a_{2n})\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Thus the column space of A is generated by the first and last columns of A. That is, $c(A) = 2$.

Case 2. A has no zero entries. Multiplying by diagonal matrices and permuting rows and/or columns, we have that

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & a_{22} & \cdots & a_{2n} \end{bmatrix} \quad \text{with} \quad 1 \leqslant a_{22} \leqslant \cdots \leqslant a_{2n}.$$

Since $r(A) = 2$, $1 < a_{2n}$. Now for $2 \leqslant i \leqslant n-1$,

$$\begin{bmatrix} 1 \\ a_{2i} \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ a_{2n} \end{bmatrix}$$

for $x = (a_{2n} - a_{2i})/(a_{2n} - 1)$ and $y = (a_{2i} - 1)/(a_{2n} - 1)$. Since x and y are in $\mathbb{F}^+$, $c(A) = 2$. Thus if $\min(m, n) = 2$ then $\mu(\mathbb{F}^+, m, n) = 2$.

Let $\rho(A)$ denote the field rank of A, that is, the rank of A in $\mathcal{M}_{m,n}(\mathbb{F})$. Since $\mathbb{F}^+$ is a subsemiring of $\mathbb{F}$, $\rho(A) \leqslant r(A)$ for all $A \in \mathcal{M}_{m,n}(\mathbb{F}^+)$.

If $\min(m, n) \geqslant 3$ and $r(A) = 2$, let A^i, A^j, and A^k be any three columns of A. Since $\rho(A) \leqslant 2$, there are scalars α, β, and γ in $\mathbb{F}$, not all zero, such that $\alpha A^i + \beta A^j + \gamma A^k = \mathbf{0}$. Since all the entries in A are nonnegative, at least one of α, β, γ is positive and one negative. We may assume that two are positive (or at least nonnegative) and one negative, say γ is negative. Then $(\alpha/-\gamma)A^i + (\beta/-\gamma)A^j = A^k$. Thus $c(A) \leqslant 2$, by Lemma 3.3. It now follows that $\mu(\mathbb{F}^+, m, n) \geqslant 2$. Thus if $m \geqslant 3$ and $n \geqslant 4$, $\mu(\mathbb{F}^+, m, n) = 2$ by Lemma 3.4.

Since $r(A) \leqslant c(A)$, it follows that $\mu(\mathbb{F}^+, m, 3) = 3$ for all $m \geqslant 3$. ■

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada under grant A4041.

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Received 20 October 1986; revised May 1987